How Hard is Control in Multi-Peaked Elections: a Parameterized Study

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ABSTRACT

We study the complexity of voting control problems in multi-peaked elections. In particular, we focus on the constructive/destructive control by adding/deleting votes under Condorcet, Maximin and Copeland$^\alpha$ voting systems. We show that the $\mathcal{NP}$-hardness of these problems (except for the destructive control by adding/deleting votes under Condorcet, which is polynomial time solvable in the general case) hold even in κ-peaked elections with κ being a very small constant. Furthermore, from the parameterized complexity point of view, our reductions actually show that these problems are $\mathcal{W}[1]$-hard in κ-peaked elections with $\kappa = 3, 4$, with respect to the number of added/deleted votes.

Categories and Subject Descriptors

F.2 [**Theory of Computation**]: Analysis of Algorithms and Problem Complexity; G.2.1 [**Combinatorics**]: Combinatorial algorithms; J.4 [**Computer Applications**]: Social Choice and Behavioral Sciences

General Terms

Algorithms

Keywords

single-peaked election, parameterized complexity, multi-peaked election, Condorcet, Maximin, Copeland

1. INTRODUCTION

Voting is a common method for preference aggregation and collective decision-making, and has applications in political elections, multiagent systems, web spam reduction, etc. For instance, in multiagent systems, it is often necessary for a group of agents to make a collective decision by means of voting in order to reach a joint goal (see [5] for a detailed discussion). Unfortunately, by Arrow's impossibility theorem [1], there is no voting system which satisfies a certain set of desirable criteria (see [1] for the details) when more than two candidates are involved. One possible way to bypass Arrow's impossibility theorem is to restrict the domain of the preferences, for instance, the single-peaked domain introduced by Black [2]. Intuitively, in a single-peaked election, one can order the candidates from left to right such that every voter's preferences increase first and then decrease after some point as the candidates are considered from left to right.

Recently, the complexity of various voting problems in single-peaked elections has been attracting attention of many researchers from both theoretical computer science and social choice communities [3, 12, 16, 18, 24]. It turned out that many voting problems being $\mathcal{NP}$-hard in general become polynomial-time solvable when restricted to single-peaked elections [3, 16]. However, most elections in practice are not purely single-peaked, which motivates researchers to study more general models of elections. We refer readers to [4, 6, 7, 10, 14] for some variants of single-peaked model. Recent complexity studies of voting problems in these variants of single-peaked model can be found in [6, 14, 25, 27].

In this paper, we consider a natural generalization of single-peaked elections, where more than one peak may occur in each vote. This generalization might be relevant for many real-world applications. For example, consider a group of people who are willing to select a special day for an event. In this setting, each voter may have several special days which he/she prefers for some reason, and the longer the other days away from these favorite days, the less they are preferred by the voter. Recently Egan [9] has discussed 2-peaked political elections in detail. We call this generalization κ-peaked elections, where at most κ peaks are allowed in each vote with respect to a given order over all candidates.

We mainly study control problems for Condorcet, Copeland$^\alpha$ and Maximin voting restricted to κ-peaked elections. In a control attack, there is an external agent (e.g., the chairman in an election) who is willing to influence the results of the election by doing some tricks. There would be two goals that the external agent wants to reach. One goal is to make some distinguished candidate win the election. The other goal would be to make someone lose the election. The former case is called a *constructive control* and the latter case is called a *destructive control*. Moreover, the tricks involved in a control attack include adding some new, unregistered votes to the registered votes, or deleting votes from the registered votes. We refer readers to [11, 19, 20] for further information on control attacks.

In the general case (the domain of the preferences is not restricted), both the constructive control and the destructive control by adding/deleting votes are $\mathcal{NP}$-hard for Copeland$^\alpha$ for every $0 \leq \alpha \leq 1$ and Maximin [13, 15]. Concerning the Condorcet voting, the constructive control by adding/deleting votes is $\mathcal{NP}$-hard

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while the destructive control by adding/deleting votes is polynomial time solvable in the general case [20]. In contrast, most of these problems become polynomial time solvable when restricted to single-peaked elections [3].

Motivated by the $\mathcal{NP}$-completeness in the general case and the polynomial-time solvability in the single-peaked case, we study the complexity of control problems for Condorcet, Copeland$^\alpha$ and Maximin voting in κ-peaked elections with respect to various values of κ, aiming at exploring the complexity border for these control problems. In particular, we prove that both the constructive and the destructive control by adding votes for Maximin and Copeland$^\alpha$ for all $0 \leq \alpha \leq 1$, and the constructive control by adding votes for Condorcet are $\mathcal{NP}$-hard in 3-peaked elections. Moreover, both the constructive control and the destructive control by deleting votes for Maximin and Copeland$^\alpha$ for all $0 \leq \alpha \leq 1$, and the constructive control by deleting votes for Condorcet are $\mathcal{NP}$-hard in 4-peaked elections. In fact, our reductions imply more stronger results. Precisely, our reductions actually show that all these $\mathcal{NP}$-hard problems in 3,4-peaked elections are $\mathcal{W}[1]$-hard, with respect to the number of added/deleted votes (see Section 2 for the discussion of parameterized complexity). See Table 1 for a summary of our main results.

Parameterized complexity of voting control problems have been extensively studied recently (See Section 2 for further details on parameterized complexity). In particular, Liu and Zhu [22] proved that both the constructive control and the destructive control by adding/deleting votes for Maximin are $\mathcal{W}[1]$-hard in the general case, with respect to the number of added/deleted votes. Moreover, Liu et al. [21] proved that the constructive control by adding/deleting votes for Condorcet is $\mathcal{W}[1]$-hard in the general case, with respect to the number of added/deleted votes. However, their reductions do not apply to 3,4-peaked elections. In fact, to show the $\mathcal{W}[1]$-hardness of the control problems in 3,4-peaked elections, we use technically completely different strategies. Liu and Zhu also studied parameterized complexity of other voting problems (see [23]). Recently, Yang and Guo [26] has also studied the complexity of control problems in κ-peaked elections. However, they considered only the r-approval voting systems. We complement their work by investigating the Condorcet, Maximin and Copeland$^\alpha$ voting. A special case of 2-peaked elections, called swoon-SP elections, were studied by Faliszewski et al. [14]. Yang and Guo [27] studied complexity of control problems in elections with bounded single-peaked width. Our work shares the same motivation of theirs—exploring the complexity border of the control problems between single-peaked elections and general elections. Therefore, our work can be also considered as a complement to theirs.

2. PRELIMINARIES

Elections: An *election* is a tuple $\mathcal{E} = (\mathcal{C}, \Pi_\mathcal{V})$, where $\mathcal{C}$ is a set of candidates and $\Pi_\mathcal{V}$ is a multiset of votes casted by a set of voters $\mathcal{V}$. Each vote is defined as a linear order $\succ$ (to represent a voter's preference) over $\mathcal{C}$. For two candidates c, c' and a vote $\succ$, we say c is *ranked above* c' if $c \succ c'$. We use $N_\mathcal{E}(c, c')$ to denote the number of votes ranking c above c' in $\mathcal{E}$. We drop the index $\mathcal{E}$ when it is clear from the context. We say c *beats* c' if $N(c, c') > N(c', c)$, and c ties c' if $N(c, c') = N(c', c)$. Moreover, the *position* of a candidate c in a vote $\succ$ is defined as $|\{c' \in \mathcal{C} \mid c' \succ c\}| + 1$. A *voting correspondence*[1] φ is a function that maps an election $\mathcal{E} =$

[1]A related terminology is *voting rule* which is defined as a function mapping an election to a single candidate. A voting correspondence can be easily modified to a voting rule using a certain tie-breaking method.

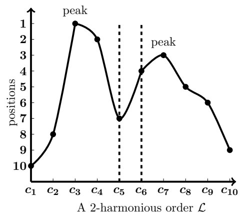

Figure 1: A 2-peaked vote $(c_3, c_4, c_7, c_6, c_8, c_9, c_5, c_2, c_{10}, c_1)$ with respect to the 2-harmonious order $\mathcal{L} = (c_1, c_2, \ldots, c_{10})$. Here, $\mathcal{L}$ is partitioned into L_1 and L_2 with $L_1 = (c_1, c_2, c_3, c_4, c_5)$ and $L_2 = (c_6, c_7, c_8, c_9, c_{10})$.

$(\mathcal{C}, \Pi_\mathcal{V})$ to a non-empty subset $\varphi(\mathcal{E})$ of $\mathcal{C}$. We call the elements in $\varphi(\mathcal{E})$ the *winners* of $\mathcal{E}$. If $\varphi(\mathcal{E})$ contains only one winner, we call it a *unique winner*; otherwise, we call them *nonunique winners*.

For simplicity, we also use $(a_1, a_2, ..., a_n)$ to denote the linear order $a_1 \succ a_2 \succ, ..., \succ a_n$. For a vote $\succ$ and a subset $C \subseteq \mathcal{C}$, let $\succ (C)$ denote the *partial vote* of $\succ$ restricted to C. For example, for a vote $\succ$ defined as (a, b, c, d, e), we have that $\succ (\{b, d, e\}) = (b, d, e)$.

Single-peaked/κ-peaked elections. An election $(\mathcal{C}, \Pi_\mathcal{V})$ is *single-peaked* if there is a linear order $\mathcal{L}$ of $\mathcal{C}$ such that for every vote $\succ_v$ in $\Pi_\mathcal{V}$ and every three candidates $a, b, c \in \mathcal{C}$ with $a \ \mathcal{L} \ b \ \mathcal{L} \ c$ or $c \ \mathcal{L} \ b \ \mathcal{L} \ a$, $c \succ_v b$ implies $b \succ_v a$, where $a \ \mathcal{L} \ b$ means a is ordered before b in $\mathcal{L}$. The candidate ordered in the first position of $\succ_v$ is the *peak* of $\succ_v$ with respect to $\mathcal{L}$.

For an order $\mathcal{L} = (c_1, c_2, \ldots, c_m)$ of $\mathcal{C}$ and a vote $\succ_v$, we say $\succ_v$ is *κ-peaked* with respect to $\mathcal{L}$, if there is a κ'-partition $L_1 = (c_1, c_2, ..., c_i)$, $L_2 = (c_{i+1}, c_{i+2}, ..., c_{i+j}), \ldots, L_{\kappa'} = (c_t, c_{t+1}, ..., c_m)$ of $\mathcal{L}$ such that $\kappa' \leq \kappa$ and $\succ_v (\mathcal{C}(L_x))$ is single-peaked with respect to L_x for all $1 \leq x \leq \kappa'$, where $\mathcal{C}(L_x)$ is the set of candidates appearing in L_x. See Fig. 1 for an example.

An election is κ-peaked if there is an order $\mathcal{L}$ of $\mathcal{C}$ such that every vote in the election is κ-peaked with respect to $\mathcal{L}$. Here $\mathcal{L}$ is called a *κ-harmonious order*.

(Weak) Condorcet Winner: A *Condorcet winner* is a candidate which beats every other candidate. An election has either no Condorcet winner or only one Condorcet winner. A *weak Condorcet winner* is a candidate which is not beaten by any other candidate.

Voting Correspondences: We mainly study the following voting correspondences.

Copeland$^\alpha$ ($0 \leq \alpha \leq 1$): For a candidate c, let $B(c)$ be the set of candidates who are beaten by c and $T(c)$ the set of candidates who tie with c. The Copeland$^\alpha$ score of c is then defined as $|B(c)| + \alpha \cdot |T(c)|$. A Copeland$^\alpha$ winner is a candidate with the highest score.

Maximin: For a candidate c, the Maximin score of c is defined as $\min_{c' \in \mathcal{C} \setminus \{c\}} N(c, c')$. A Maximin winner is a candidate with the highest Maximin score.

Problem Definitions: Problems studied here are characterized by four factors, CC|DC specifying constructive or destructive control, AV|DV specifying adding or deleting votes, φ specifying the voting correspondence, and UNI|NON specifying the unique-winner or nonunique-winner models. For example, CCAV-φ-UNI denotes the problem of constructive control by adding votes for the unique-winner model under the voting correspondence φ. In the inputs of all these problems, we have a set $\mathcal{C}$ of candidates, a distinguished candidate p, and an integer $R \geq 0$. In the deleting votes case, there is only one multiset $\Pi_{\mathcal{V}_1}$ of (registered) votes in the input, while the adding votes case distinguishes two multisets of votes, $\Pi_{\mathcal{V}_1}$ the multiset of registered votes and $\Pi_{\mathcal{V}_2}$ the multiset of unregistered votes. In addition, in both cases, a κ-harmonious order $\mathcal{L}$ is given

	number of peaks κ									Evidence for	
	$\kappa = 1$	$\kappa = 3$				$\kappa \geq 4$					
	for all	CC		DC		CC		DC			
		AV	DV	AV	DV	AV	DV	AV	DV	$\kappa = 3$	$\kappa = 4$
Condorcet		$\mathcal{W}[1]$-**hard**		$\mathcal{P}$		$\mathcal{W}[1]$-**hard**		$\mathcal{P}$		Theorem 3	Theorem 6
Maximin	$\mathcal{P}$	$\mathcal{W}[1]$-**hard**	?	$\mathcal{W}[1]$-**hard**	?	$\mathcal{W}[1]$-**hard**				Theorem 1	Theorem 4
Copeland$^\alpha$	$(\alpha = 1)$	$\mathcal{W}[1]$-**hard**		$\mathcal{W}[1]$-**hard**		$\mathcal{W}[1]$-**hard**				Theorem 2	Theorem 5

Table 1: A summary of the complexity of control problems under Condorcet, Maximin and Copeland$^\alpha$ in κ-peaked elections. Here, "$\mathcal{P}$" stands for polynomial-time solvable. Our results are in bold. Moreover, our results for Copeland$^\alpha$ apply to all $0 \leq \alpha \leq 1$. The $\mathcal{W}[1]$-hardness results of the control by adding/deleting votes are with respect to the number of added/deleted votes. Note that when $\kappa = m/2 + 1$, κ-peaked elections are the general elections. The polynomial time solvability results in single-peaked elections (1-peaked elections) are from [3] (The result for Copeland$^\alpha$ only holds for $\alpha = 1$. The complexity for $0 \leq \alpha < 1$ is unknown). The polynomial time solvability of the destructive control by adding/deleting votes for Condorcet is from [20]. The entries filled with "?" means the corresponding problems are open.

in advance. Moreover, in the deleting votes case, all votes in $\Pi_{\mathcal{V}_1}$ are κ-peaked with respect $\mathcal{L}$, and in the adding votes case, all votes in both $\Pi_{\mathcal{V}_1}$ and $\Pi_{\mathcal{V}_2}$ are κ-peaked with respect to $\mathcal{L}$. The goal here is to make p win (CC) or lose (DC) the election by adding at most R unregistered votes (AV) or deleting at most R votes (DV). Strictly speaking, (weak) Condorcet is not a voting correspondence, since the winner set could be empty. However, the complexity of control problems has been widely studied for (weak) Condorcet since the seminal paper by Bartholdi et al. [20]. We also study it here due to the importance of the concept of (weak) Condorcet winner. In this case, the constructive control aims to make the distinguished candidate the Condorcet winner (UNI) or a weak Conodrcet winner (NON), while the destructive control aims to prevent the distinguished candidate from being the Condorcet winner (UNI) or a weak Condorcet winner (NON).

Parameterized Complexity: A *parameterized problem* is a language $L \subseteq \Sigma^* \times \mathbb{N}$, where Σ is a finite alphabet. The first component $I \in \Sigma$ is called the *main part* of the problem while the second component $k \in \mathbb{N}$ is called the *parameter*. Downey and Fellows [8] established the parameterized complexity theory, where the class $\mathcal{FPT}$ (stands for fixed-parameter tractable) includes all parameterized problems which admit $O(f(k) \cdot |I|^{O(1)})$-time algorithms. Here $f(k)$ is a computable function. Another important parameterized complexity class is $\mathcal{W}[1]$ which is the basic class for showing fixed-parameter intractability results. A problem is $\mathcal{W}[1]$-hard if all problems in $\mathcal{W}[1]$ are $\mathcal{FPT}$-reducible to the problem. We can show a problem being $\mathcal{W}[1]$-hard by giving an $\mathcal{FPT}$-reduction from another $\mathcal{W}[1]$-hard problem.

Given two parameterized problems Q and Q', an $\mathcal{FPT}$*-reduction* from Q to Q' is an algorithm that takes as input an instance (I, k) of Q and outputs an instance (I', k') of Q' such that

(1) the algorithm runs in $f(k) \cdot |I|^{O(1)}$ time; and

(2) $(I, k) \in Q$ if and only if $(I', k') \in Q'$; and

(3) $k' \leq g(k)$, where g is a computable function in k.

All $\mathcal{W}[1]$-hardness reductions in this paper are from the INDEPENDENT SET in 2-interval graphs which is $\mathcal{W}[1]$-hard (and also $\mathcal{NP}$-hard) [17].

A *2-interval* I is a union of 2 disjoint intervals of the real line. We write $I = \{I^1, I^2\}$ with I^1, I^2 disjoint intervals, and $I = I^1 \cup I^2$. The endpoints of I is the endpoints of I^1 union the endpoints of I^2. Given a pair of 2-intervals $I = \{I^1, I^2\}$ and $J = \{J^1, J^2\}$, these two 2-intervals intersect if they share a common point, i.e. if $(I^1 \cup I^2) \cap (J^1 \cup J^2) \neq \emptyset$.

INDEPENDENT SET in 2-interval graphs
Input: A collection $\mathcal{I} = \{I_1, I_2, ..., I_n\}$ of 2-intervals.
Parameter: An integer $k \geq 0$.
Question: Is there a subcollection $\mathcal{I}' \subseteq \mathcal{I}$ of size k so that no two 2-intervals in $\mathcal{I}'$ intersect?

Remarks: All the $\mathcal{FPT}$-reductions in this paper take polynomial time in both the main part and the parameter. Therefore, all the $\mathcal{FPT}$-reductions in this paper are also polynomial-time many-one reductions, implying that all $\mathcal{W}[1]$-hard problems shown in this paper are also $\mathcal{NP}$-hard. However, for ease of exposition, we only explicitly state the $\mathcal{W}[1]$-hardness results in the theorems.

3. 3-PEAKED ELECTIONS

This section studies control problems in 3-peaked elections under Maximin, Condorcet and Copeland$^\alpha$ voting. We first examine the Maximin voting. It is known that both the constructive and the destructive control by adding votes are $\mathcal{NP}$-hard for Maximin in general [13]. The following theorem shows that both $\mathcal{NP}$-hardness hold even in 3-peaked elections. In fact, from the parameterized complexity point of view, we prove that both problems are $\mathcal{W}[1]$-hard with respect to the number of added votes. The following notations will be used in all $\mathcal{FPT}$-reductions in this paper.

Let ( ) denote an empty order containing no element. For a linear order $A = (a_1, \ldots, a_n)$, let $A[a_i, a_j]$ (resp. $A(a_i, a_j]$, $A[a_i, a_j)$ and $A(a_i, a_j)$) with $i \leq j$ be the suborder $(a_i, a_{i+1}, \ldots, a_j)$ (resp. $(a_{i+1}, a_{i+2}, \ldots, a_j)$ if $i < j$ and () if $i = j$, $(a_i, a_{i+1}, \ldots, a_{j-1})$ if $i < j$ and () if $i = j$, and $(a_{i+1}, a_{i+2}, \ldots, a_{j-1})$ if $i < j - 1$ and () if $j \geq i \geq j - 1$), and let $A[a_j, a_i]$ (resp. $A[a_j, a_i)$, $A(a_j, a_i]$ and $A(a_j, a_i)$) be the reversed order of $A[a_i, a_j]$ (resp. $A(a_i, a_j]$, $A[a_i, a_j)$ and $A(a_i, a_j)$). For two linear orders $A = (a_1, a_2, ..., a_n)$ and $B = (b_1, b_2, ..., b_m)$ with $A \cap B = \emptyset$, denote by (A, B) the linear order $(a_1, a_2, ..., a_n, b_1, b_2, ..., b_m)$.

THEOREM 1. *DCAV-Maximin-UNI, DCAV-Maximin-NON, CCAV-Maximin-UNI and CCAV-Maximin-NON in 3-peaked elections are all $\mathcal{W}[1]$-hard with respect to the number of votes added.*

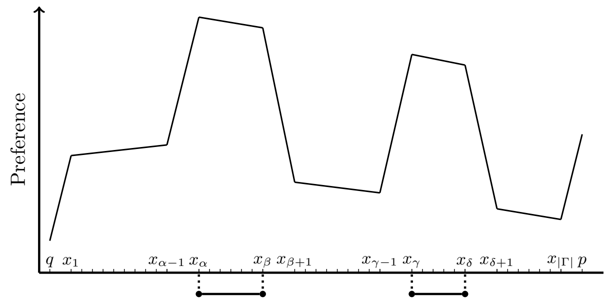

Figure 2: An unregistered vote corresponding to a 2-interval.

PROOF. We first prove the $\mathcal{W}[1]$-hardness for DCAV-Maximin-UNI in 3-peaked elections by an $\mathcal{FPT}$-reduction from INDEPENDENT SET on 2-interval graphs. For a given instance $\mathcal{F} = (\mathcal{I} = (I_1, I_2, ..., I_n), k)$ of the INDEPENDENT SET problem on 2-interval graphs, we construct an instance for DCAV-Maximin-UNI in 3-peaked elections as follows. We denote by I_i^1 and I_i^2 the two intervals of I_i. Let $D(I_i)$ be the endpoints of I_i, and let $\Gamma = \cup_{i\in[n]} D(I_i)$. Moreover, let $\vec{\Gamma} = (x_1, x_2, ..., x_{|\Gamma|})$ be the order of Γ with $x_i < x_{i+1}$ for all $i \in [|\Gamma| - 1]$.

Candidates: $\mathcal{C} = \Gamma \cup \{p, q\}$ where q is the distinguished candidate. Concretely, for each $x \in \Gamma$, we create a candidate. For ease of exposition, we still use x to denote the corresponding candidate of the endpoint x.

3-Harmonious Order: $\mathcal{L} = (q, \vec{\Gamma}, p)$.

Registered Votes: We create $3k - 1$ registered votes in total. Concretely, we create $2k - 1$ registered votes defined as $\mathcal{L}[q, p]$, and k registered votes defined as $(p, \mathcal{L}[q, x_{|\Gamma|}])$. The comparisons between every two candidates are summarized in Table 3. Let $\mathcal{E}$ be the election with the registered votes.

	p	q	$x_j (i < j)$	$x_j (i > j)$
p	-	k	k	
q	$2k - 1$	-	$3k - 1$	
x_i	$2k - 1$	0	$3k - 1$	0

Table 2: Comparisons between candidates in the $\mathcal{W}[1]$-hardness proof for DCAV-Maximin-UNI in Theorem 1. Each entry with row indicated by candidate c and column indicated by candidate c' is $N(c, c')$, the number of registered votes ranking c above c'. Here, $N(\cdot)$ is based on the registered votes.

Unregistered Votes: The unregistered votes are created according to the intervals in $\mathcal{F}$. Precisely, for every 2-interval $I_i = \{I_i^1, I_i^2\}$, we create an unregistered vote. Let x_α and x_β with $x_\alpha \leq x_\beta$ denote the left endpoint and the right endpoint of I_i^1 respectively, and x_γ and x_δ with $x_\gamma \leq x_\delta$ denote the left endpoint and the right endpoint of I_i^2 respectively. Without loss of generality, assume that I_i^2 is on the right side of I_i^1, that is $x_\beta < x_\gamma$. The unregistered vote $\succ_{I_i}$ corresponding to I_i is defined as

$$(\mathcal{L}[x_\alpha, x_\beta], \mathcal{L}[x_\gamma, x_\delta], p, \mathcal{L}(x_\alpha, x_1], \mathcal{L}(x_\beta, x_\gamma), \mathcal{L}(x_\delta, x_{|\Gamma|}], q).$$

See Fig. 2 for an illustration.

Number of Added Votes: $R = k$.

Now we come to show the correctness of the reduction. First observe that q is the current winner with Maximin score $2k - 1$. Moreover, the Maximin score of q cannot increase by adding unregistered votes to the election, since q is ranked below every other candidate in every unregistered vote; and thus, q will have a Maximin score of $2k - 1$ points in the final election. Furthermore, every $x_i \in \Gamma$ cannot have a no less Maximin score than that of q by adding at most k votes. This is because $N(x_i, q) = 0$ with respect to the registered votes; and thus, the maximum Maximin score of every x_i can be at most k in the final election. Therefore, the only candidate which has chance to have a no less score than that of q is the candidate p.

($\Rightarrow$:) Suppose that $\mathcal{F}$ has an independent set S of size k. We claim that q is no longer the unique winner after adding all unregistered votes corresponding to S to the election $\mathcal{E}$. Let $\Pi_S = \{\succ_I | I \in S\}$ be the set of the unregistered votes corresponding to S, and $\mathcal{E}'$ be the final election obtained from $\mathcal{E}$ by adding Π_S. Due to the construction of the unregistered votes, for every $x_i \in \Gamma$ there is at most one vote in Π_S which ranks x_i above p. This implies that $N_{\mathcal{E}'}(p, x_i) \geq 2k-1$ for every $x_i \in \Gamma$. Moreover, since p is ranked above q in every unregistered vote, we have that $N_{\mathcal{E}'}(p, q) = 2k$. It is now easy to see that q is no longer the unique winner.

($\Leftarrow$:) Suppose that q is not the unique winner after adding at most k unregistered votes. Let Π_S be the unregistered votes added to the election $\mathcal{E}$. Due to the above analysis, we know that p is the candidate which prevents q from being the unique winner in the final election. Since q has a Maximin score $2k - 1$ in $\mathcal{E}$, for every candidate $x_i \in \Gamma$ there has to be at least $k - 1$ votes in Π_S which rank p above x_i. Due to the construction of the unregistered votes, this happens only if there is an independent set of size k in $\mathcal{F}$.

The proof for DCAV-Maximin-NON is similar to the above one with the difference that we create one less registered vote defined as $\mathcal{L}[q, p]$. To prove CCAV-Maximin-NON, we adopt the same reduction as for DCAV-Maximin-UNI but with p being the distinguished candidate. To prove CCAV-Maximin-UNI, we adopt the same reduction as for DCAV-Maximin-NON and set p as the distinguished winner. □

Now we examine the Copeland control in 3-peaked elections. Recall that in the general case, both the constructive control and the destructive control by adding votes under Copeland$^\alpha$ are $\mathcal{NP}$-hard for every $0 \leq \alpha \leq 1$ [15]. In the following, we prove that both problems are not only $\mathcal{NP}$-hard but also $\mathcal{W}[1]$-hard even in 3-peaked elections, with respect to the number of added votes.

THEOREM 2. *DCAV-Copeland$^\alpha$-UNI, DCAV-Copeland$^\alpha$-NON, CCAV-Copeland$^\alpha$-UNI and CCAV-Copeland$^\alpha$-NON in 3-peaked elections are all $\mathcal{W}[1]$-hard for every $0 \leq \alpha \leq 1$, with respect to the number of added votes.*

PROOF. We first show the proof for DCAV-Copeland^0-NON in 3-peaked elections from an $\mathcal{FPT}$-reduction from the INDEPENDENT SET problem on 2-interval graphs. Given an instance $\mathcal{F} = (\mathcal{I} = (I_1, I_2, ..., I_n), k)$ of the INDEPENDENT SET problem on 2-interval graphs, we construct an instance for DCAV-Copeland^0-NON in 3-peaked elections as follows. The notations $I_i^1, I_i^2, D(I_i), \Gamma$ and $\vec{\Gamma}$ hereinafter are defined in the same way as in the proof of Theorem 1.

Candidates: $\mathcal{C} = \Gamma \cup \{p, q, y\}$ where q is the distinguished candidate.

3-Harmonious Order: $\mathcal{L} = (q, \vec{\Gamma}, p, y)$.

Registered Votes: We create $3k - 3$ registered votes in total. Concretely, we create $2k-3$ registered votes defined as $(q, y, \mathcal{L}[x_1, p]$, and k registered votes defined as $(p, q, y, \mathcal{L}[x_1, x_{|\Gamma|}])$. It is easy to verify that q is a Copeland^0 winner (precisely, q is the current unique winner). Let $\mathcal{E}$ be the election with the registered votes. The comparisons are summarized in Table 3.

Unregistered Votes: The unregistered votes are created according to the intervals in $\mathcal{F}$. Precisely, for every 2-interval $I_i =$

	p	q	$x_j(i<j)$	$x_j(i>j)$	y
p	-	k	k		
q	$2k-3$	-	$3k-3$		
x_i	$2k-3$	0	$3k-3$	0	
y	$2k-3$	0	$3k-3$		-

Table 3: Comparisons between candidates in the proof for DCAV-Copeland^0-NON in Theorem 2. Each entry with row indicated by candidate c and column indicated by candidate c' is $N(c,c')$, the number of registered votes ranking c above c'. Here, $N(\cdot)$ is based on the registered votes.

$\{I_i^1, I_i^2\}$, we create an unregistered vote as follows. Let x_α and x_β with $x_\alpha \leq x_\beta$ denote the left endpoint and the right endpoint of I_i^1 respectively, and x_γ and x_δ with $x_\gamma \leq x_\delta$ denote the left endpoint and the right endpoint of I_i^2 respectively. Without loss of generality, assume that I_i^2 is on the right side of I_i^1, that is, $x_\beta < x_\gamma$. The unregistered vote $\succ_{I_i}$ corresponding to I_i is defined as

$$(\mathcal{L}[x_\alpha, x_\beta], \mathcal{L}[x_\gamma, x_\delta], p, y, \mathcal{L}(x_\alpha, x_1], \mathcal{L}(x_\beta, x_\gamma), \mathcal{L}(x_\delta, x_{|\Gamma|}], q).$$

Number of Added Votes: $R = k$.

Now we show the correctness argument.

($\Rightarrow$:) Suppose that $\mathcal{F}$ has an independent set S of size k. Consider the election $\mathcal{E}'$ after adding all the unregistered votes $\Pi_S = \{\succ_I | \ I \in S\}$ corresponding to S to $\mathcal{E}$. Clearly, we have $4k-3$ votes in $\mathcal{E}'$. Since p is ranked above q and y in every unregistered vote, p beats both q and y in $\mathcal{E}'$. Due to the construction of the unregistered votes and the fact that S is an independent set, for every $x_i \in \Gamma$ there is at most one unregistered vote in Π_S which ranks x_i above p. Therefore, p also beats every x_i in $\mathcal{E}'$ by at least $2k-1$. Summary all above, p beats every other candidate in $\mathcal{E}'$, implying that q is no longer a Copeland^0 winner.

($\Leftarrow$:) Suppose that Π_S is the multiset of unregistered votes added to the election which makes q no longer a Copeland^0 winner. Let $\mathcal{E}'$ be the final election. We claim that p is the only candidate which can prevent q from being a winner in $\mathcal{E}'$. Observe first that the candidate q beats y and every x_i in $\mathcal{E}'$. In this case, in order to prevent q from being a Copeland^0 winner, q has to be beaten by p in $\mathcal{E}'$. However, once q is beaten by p in $\mathcal{E}'$, y is also beaten by p in $\mathcal{E}'$. Hence, y cannot have a no less Copeland^0 score than that of q. In addition, since every $x_i \in \Gamma$ is beaten by y and q in $\mathcal{E}'$, no one in Γ can prevent q from being a winner in $\mathcal{E}'$. Therefore, the above claim holds. Since q beats every other candidate except p, in order to make p have a no less score than that of q, p has to beat every other candidate. This happens only if Π_S contains k unregistered votes, and moreover, for every candidate x_i there is at most one vote in Π_S ranking x_i above p. The latter condition directly imply that the set of 2-intervals corresponding to Π_S forms an independent set, and the former condition implies that $|S| = k$. The proof for DCAV-Copeland^0-NON is finished.

The above reduction does not apply to DCAV-Copeland^0-UNI directly, since, in this case, q could also become not a unique-winner when there is no independent set of size k for $\mathcal{E}$. To check this, consider the situation where p is beaten by some $x \in \Gamma$ (but p beats every other candidate in Γ) in $\mathcal{E}'$. This can happen when we add two unregistered votes corresponding to two 2-intervals which intersect only at x to the election. In this situation, p beats every other candidate except x and q beats every other candidate except p, implying that q is no longer a unique winner. In order to overcome this drawback for DCAV-Copeland^0-UNI, we need to restrict the 2-intervals in $\mathcal{E}$ in such a way that once two 2-intervals intersect, they do not intersect at only one point. See Fig. 3 for an illustration. This restriction does not change the $\mathcal{W}[1]$-hardness of the INDEPENDENT SET problem on 2-interval graphs [17]. Under this restriction, once two unregistered votes corresponding to two 2-intervals that intersect are added to the election, p will be beaten by at least two candidates in Γ, implying that p cannot prevent q from being the unique Copeland^0 winner if there is no independent set of size k for $\mathcal{F}$.

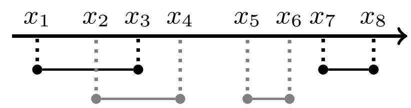

Figure 3: An illustration of the restriction on the 2-intervals in the $\mathcal{W}[1]$-hardness proof for DCAV-Copeland^0-UNI in Theorem 2. Once two 2-intervals intersect, they intersect at more than one point (in a continuity interval). Therefore, for each intersection of two 2-intervals, we create at least two candidates.

The reductions for the Copeland$^\alpha$ control with $0 < \alpha \leq 1$ are similar to the above ones with the difference that we create further polynomially many dummy candidates in every intersection of two 2-intervals. $\square$

Now we come to the Condorcet voting. The following theorem summarizes our findings. Recall that in the constructive control for Condorcet, the unique-winner model aims to make the distinguished candidate the Condorcet winner, while the non-unique winner model aims to make p a weak Condorcet winner. In general, the constructive control by adding votes under Condorcet is $\mathcal{NP}$-hard, while the destructive case turned out to be polynomial time solvable. [20].

THEOREM 3. *CCAV-Condorcet-UNI and CCAV-Condorcet-NON in 3-peaked elections are $\mathcal{W}[1]$-hard with respect to the number of added votes.*

PROOF. The proof for CCAV-Condorcet-UNI is exactly the same as for CCAV-Copeland^0-UNI. The proof for CCAV-Condorcet-NON is similar to CCAV-Condorcet-UNI with the difference that we create one more registered votes defined as $(q, y, \mathcal{L}[x_1, p])$. $\square$

4. 4-PEAKED ELECTIONS

In the previous subsection, we have discussed control by adding votes in 3-peaked elections. In this section, we consider control by deleting votes in 4-peaked elections for Condorcet, Copeland and Maximin voting. We first examine the Maximin voting. It is known that both the constructive control and the destructive control by deleting votes are $\mathcal{NP}$-hard for Maximin in general [13]. The following theorem shows both problems are $\mathcal{W}[1]$-hard even in 4-peaked elections, with respect to the number of added votes.

THEOREM 4. *CCDV-Maximin-UNI, CCDV-Maximin-NON, DCDV-Maximin-UNI and DCDV-Maximin-NON are $\mathcal{W}[1]$-hard in 4-peaked elections, with respect to the number of deleted votes.*

PROOF. Our reductions are again from INDEPENDENT SET on 2-interval graphs. We adopt another restriction on the 2-intervals (different from the one in the proof of Theorem 2). For two 2-intervals I and J, we say I covers J if $J \subseteq I$. See Figure 4 for an illustration. The given instances are then restricted in a way so that there are no two 2-intervals with one covers the other. This restriction does not change the $\mathcal{W}[1]$-hardness of the problem [17].

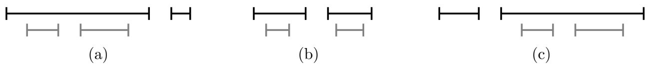

Figure 4: This figure shows all the three different ways of how two 2-intervals intersect. The two 2-intervals (black and gray) are draw at different levels for the sake of clarity. Nevertheless, they are both defined on the real line.

	p	q	$x_j(j>i)$	$x_j(j<i)$
p	-	$2n-k+2$	$2n-k+2$	
q	$2n$	-	$2n+2$	
x_i	$2n$	$2n-k$	$\times$	$\times$

Table 4: Comparisons between candidates in the $\mathcal{W}[1]$-hardness proof for CCDV-Maximin-UNI and DCDV-Maximin-NON in Theorem 4. The comparisons between x_i and x_j are marked with '$\times$' since they cannot be exactly determined. However, the comparisons do not play any role in the correctness proof.

Let $\mathcal{F} = (\mathcal{I} = (I_1, I_2, ..., I_n), k)$ be a given instance of the INDEPENDENT SET problem on 2-interval graphs. The following reduction applies to both CCDV-Maximin-UNI and DCDV-Maximin-NON. We will discuss the constructions for the other two problems latter. In the following, $I_i^1, I_i^2, D(I_i), \Gamma = \cup_i D(I_i)$ and $\vec{\Gamma} = (x_1, x_2, ..., x_{|\Gamma|})$ are defined in the same way as in the proof of Theorem 1.

Candidates: $\mathcal{C} = \Gamma \cup \{p, q, x_0\}$, where p is the distinguished candidate in CCDV-Maximin-UNI, and q is the distinguished candidate in DCDV-Maximin-NON.

4-Harmonious Order: $\mathcal{L} = (q, x_0, \vec{\Gamma}, p)$.

Votes: We first create n votes defined as $(\mathcal{L}[x_0, p], q)$, $n-k$ votes defined as $(\mathcal{L}[p, x_0], q)$ and 2 votes defined as $(p, q, \mathcal{L}[x_0, x_{|\Gamma|}])$. Then we create $2n$ votes corresponding to the 2-intervals in $\mathcal{F}$. Precisely, for every 2-interval $I_i = \{I_i^1, I_i^2\}$, we create two votes $\succ_{I_i^1}$ and $\succ_{I_i^2}$ as follows. Let x_α and x_β with $x_\alpha \leq x_\beta$ denote the left endpoint and the right endpoint of I_i^1, respectively, and x_γ and x_δ with $x_\gamma \leq x_\delta$ denote the left endpoint and the right endpoint of I_i^2, respectively. Without loss of generality, assume that I_i^2 is on the right side of I_i^1, that is, $x_\beta < x_\gamma$. Then, $\succ_{I_i^1}$ and $\succ_{I_i^2}$ are respectively defined as

$$(\mathcal{L}[q, x_{\alpha-1}], \mathcal{L}(x_\beta, x_\gamma), \mathcal{L}(x_\delta, p], \mathcal{L}[x_\alpha, x_\beta], \mathcal{L}[x_\gamma, x_\delta]) \text{ and}$$

$$(q, \mathcal{L}[x_\alpha, x_\beta], \mathcal{L}[x_\gamma, x_\delta], p, \mathcal{L}[x_0, x_{\alpha-1}], \mathcal{L}(x_\beta, x_\gamma), \mathcal{L}(x_\delta, x_{|\Gamma|}]).$$

In the following, let $\Pi_1 = \{\succ_{I_i^1} | \ i = 1, 2, ..., n\}$ and $\Pi_2 = \{\succ_{I_i^2} | \ i = 1, 2, ..., n\}$. Let $\mathcal{E}$ be the election with the above $4n-k+2$ votes. The comparisons are summarized in Table 4.

Number of deleted votes: $R = k$.

It is clear that q has the maximum Maximin score $2n$ and is thus the unique Maximin winner. We now show the correctness for the CCDV-Maximin-UNI.

($\Rightarrow$:) Suppose that $\mathcal{F}$ has an independent set S of size k. We claim that we can make p the unique Maximin winner by deleting votes corresponding to S in Π_1. Let $\Pi_S = \{\succ_{I^1} | \ I \in S\}$ be the set of the votes corresponding to S in Π_1, and let $\mathcal{E}'$ be the final election obtained from $\mathcal{E}$ by deleting all votes in Π_S. Due to the construction and the fact that S is an independent set, we have that for every $x_i \in \Gamma$ there is at most one vote in Π_S which ranks p above x_i. This implies that $N_{\mathcal{E}'}(p, x_i) \geq 2n-k+1$ for every $x_i \in \Gamma$. Moreover, since q is ranked above p in every vote in Π_S, $N_{\mathcal{E}'}(p, q) = 2n-k+2$ and $N_{\mathcal{E}'}(q, p) = 2n-k$. Therefore, the Maximin score of p is at least $2n-k+1$ while the Maximin score of q is at most $2n-k$. Finally, since q is ranked above every $x_i \in \Gamma$ in every vote in Π_S, we have that $N_{\mathcal{E}'}(x_i, q) = 2n-k$, implying that every x_i has a Maximin score at most $2n-k$. Summary all above, we know that p is the unique winner in the final election $\mathcal{E}'$.

($\Leftarrow$:) Suppose that p becomes the unique winner after deleting at most k votes. Let Π_S be the set of the votes that are deleted. Let $\mathcal{E}'$ be the final election obtained from $\mathcal{E}$ by deleting all votes in Π_S. Observe first that Π_S contains no vote which ranks p above q, since otherwise, $N_{\mathcal{E}'}(p, q) \leq 2n-k+1$ and $N_{\mathcal{E}'}(q, p) \geq 2n-k+1$, contradicting with the fact that p is the unique winner in $\mathcal{E}'$. Since we can delete at most k votes and the Maximin score of q is $2n$ in the original election $\mathcal{E}$, the final Maximin score of q is at least $2n-k$. Since p is the unique winner in $\mathcal{E}'$, the Maximin score of p is at least $2n-k+1$ in the final election $\mathcal{E}'$. Therefore, for every $x_i \in \Gamma$ there is at most one vote in Π_S which ranks p above x_i. Due to the fact, we have the following claim.

Claim. Π_S contains no vote in Π_2.

(Proof of the Claim.) We prove the claim by contradiction. Suppose that $\succ_{I_i^2} \in \Pi_S \cap \Pi_2$ is a vote corresponding to a 2-interval I_i. Let A be the set of candidates which lie in the 2-interval I_i. Due to the construction, all the candidates in $\Gamma \setminus A$ are ranked below p. Let I_j be another 2-interval which corresponds to another vote $\succ_{I_j^u} \neq \succ_{I_i^2}$. Let B be the set of candidates which lie in the 2-interval I_j. Due to the restriction of the instance, we know that $B \setminus A \neq \emptyset$. Therefore, $u \neq 1$, since otherwise, both $\succ_{I_j^u}$ and $\succ_{I_i^2}$ rank p above every candidate in $B \setminus A$. However, u cannot be equal to 2 either, since otherwise, both $\succ_{I_j^u}$ and $\succ_{I_i^2}$ rank p above the candidate x_0. See Figure 5 for an illustration.

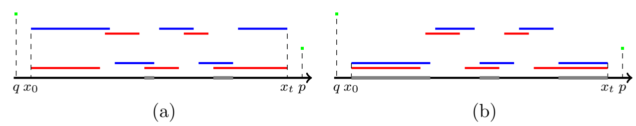

Figure 5: An illustration of the Claim in the proof of Theorem 4. Here, $t = |\Gamma|$. In both figures (a) and (b), most comparisons among the candidates in Γ are not explicitly showed. Moreover, the figure (a) shows the case that $u = 1$ and the figure (b) shows the case that $u = 2$. In either case, the candidates lie in the gray interval are ranked below p in the two votes corresponding to the red 2-interval and the blue 2-interval.

Due to the above claim, we know that $\Pi_S \subseteq \Pi_1$. Let S be the set of 2-intervals corresponding to Π_S. Since for every $x_i \in \Gamma$ there is at most one vote in Π_S which ranks p above x_i, there are no two 2-intervals in S which intersect, implying that S is an independent set for $\mathcal{F}$. It remains to show that $|S| = k$, or equivalently, $|\Pi_S| = k$. This holds, since otherwise, q would have a Maximin score at least $n-k+1$ in $\mathcal{E}'$, while p has a Maximin score at most $n-k+1$ in $\mathcal{E}'$, contradicting with the fact that p is the unique winner in $\mathcal{E}'$.

To check that the same reduction applies to DCDV-Maximin-NON, observe first that no $x_i \in \Gamma$ can have a higher Maximin score than that of q in the final election: since $N_{\mathcal{E}}(x_i, q) = 2n-k$, $N_{\mathcal{E}}(q, p) = 2n$ and we can delete at most k votes, every x_i would have a Maximin score at most $2n-k$ and q would have a Maximin score at least $2n-k$ in the final election. Due to this fact, p is the only candidate which can prevent q from being a winner. The above argument then works.

Now we discuss the reductions for CCDV-Maximin-NON and

	p	q	$x_j (j > i)$	$x_j (j < i)$
p	-	$2n-k+1$	$2n-k+1$	
q	$2n$	-	$2n+1$	
x_i	$2n$	$2n-k$	$\times$	$\times$

Table 5: Comparisons between candidates in the $\mathcal{W}[1]$-hardness proof for CCDV-Maximin-NON and DCDV-Maximin-UNI in Theorem 4. The comparisons between x_i and x_j are marked with '$\times$' since they cannot be exactly determined. However, they do not play any role in the correctness proof.

	p	q	$x_j (j > i)$	$x_j (j < i)$
p	-	$2n-k+1$	$2n-k+2$	
q	$2n$	-	$3n-k+2$	
x_i	$2n-1$	$n-1$	$\times$	$\times$

Table 6: Comparisons between candidates in the $\mathcal{W}[1]$-hardness proof for CCDV-Copeland$^\alpha$-UNI, CCDV-Copeland$^\alpha$-NON and DCDV-Copeland$^\alpha$-NON in Theorem 5. The comparisons between x_i and x_j cannot be be exactly determined. However, the comparisons do not paly any role in the correctness proof.

DCDV-Maximin-UNI. Analogously, we adopt the same reduction as for CCDV-Maximin-UNI with the following differences. First, we create only one vote defined as $(p, q, \mathcal{L}[x_0, x_{|\Gamma|}])$, instead of two as in the reduction for CCDV-Maximin-UNI. Moreover, in CCDV-Maximin-NON we set p as the distinguished candidate, while in DCDV-Maximin-UNI we set q as the distinguished candidate. The comparisons between every two candidates are shown in Table 5. $\square$

Now we study Copeland control by deleting votes in 4-peaked elections. Recall that in general, both the constructive control and the destructive control by deleting votes for Copeland$^\alpha$ are $\mathcal{NP}$-hard, for every $0 \le \alpha \le 1$ [15]. Our results are summarized in the following theorem.

THEOREM 5. *CCDV-Copeland$^\alpha$-UNI, CCDV-Copeland$^\alpha$-NON, DCDV-Copeland$^\alpha$-UNI and DCDV-Copeland$^\alpha$-NON for every $0 \le \alpha \le 1$ are $\mathcal{W}[1]$-hard in 4-peaked elections, with respect to the number of deleted votes.*

PROOF. Our reductions are again from the INDEPENDENT SET problem on 2-interval graphs. Moreover, we adopt the same restriction on 2-intervals as in the proof of theorem 2. Therefore, in the given instance, every two 2-intervals either do not intersect or intersect at more than one point. Let $\mathcal{F} = (\mathcal{I} = (I_1, I_2, ..., I_n), k)$ be the given instance of the INDEPENDENT SET problem on 2-interval graphs, we construct an instance for the problems stated in the theorem as follows. We first consider CCDV-Copeland$^\alpha$-UNI, CCDV-Copeland$^\alpha$-NON and DCDV-Copeland$^\alpha$-NON. Hereby, the notations $I_i^1, I_i^2, D(I_i), \Gamma = \cup_{i \in [n]} I_i$ and $\vec{\Gamma} = (x_1, x_2, ..., x_{|\Gamma|})$ are defined in the same way as in the proof of Theorem 1.

Candidates: $\mathcal{C} = \Gamma \cup \{p, q\}$, where p is the distinguished candidate in CCDV-Copeland$^\alpha$-UNI, CCDV-Copeland$^\alpha$-NON and q is the distinguished candidate in DCDV-Copeland$^\alpha$-NON.

4-Harmonious Order: $\mathcal{L} = (q, \vec{\Gamma}, p)$.

Votes: We first create $n-1$ votes defined as $(\mathcal{L}[x_1, p], q)$ and $n-k+2$ votes defined as $(p, \mathcal{L}[q, x_{|\Gamma|}])$. Then, we create $2n$ votes according to the 2-intervals in $\mathcal{F}$. Precisely, for every 2-interval $I_i = \{I_i^1 = [x_\alpha, x_\beta], I_i^2 = [x_\gamma, x_\delta]\}$ we create two votes $\succ_{I_i^1}$ and $\succ_{I_i^2}$ as follows. Without loss of generality, assume that I_i^2 is on the right side of I_i^1, that is, $x_\beta < x_\gamma$. Then, $\succ_{I_i^1}$ and $\succ_{I_i^2}$ are respectively defined as

$$(\mathcal{L}[q, x_\alpha), \mathcal{L}(x_\beta, x_\gamma), \mathcal{L}(x_\delta, p], \mathcal{L}[x_\alpha, x_\beta], \mathcal{L}[x_\gamma, x_\delta]) \text{ and}$$

$$(q, \mathcal{L}[x_\alpha, x_\beta], \mathcal{L}[x_\gamma, x_\delta], p, \mathcal{L}[x_1, x_\alpha), \mathcal{L}(x_\beta, x_\gamma), \mathcal{L}(x_\delta, x_{|\Gamma|}]).$$

In the following, let $\Pi_1 = \{\succ_{I_i^1} \mid i = 1, 2, ..., n\}$ and $\Pi_2 = \{\succ_{I_i^2} \mid i = 1, 2, ..., n\}$. Let $\mathcal{E}$ be the election with the above $4n - k + 1$ votes. The comparisons are shown in Table 6.

Number of deleted votes: $R = k$.

Now we show the correctness argument for CCDV-Copeland$^\alpha$-UNI.

($\Rightarrow$:) Suppose that $\mathcal{F}$ has an independent set S of size k. We claim that p is the unique Copeland$^\alpha$ winner after deleting votes corresponding to S in Π_1. Let $\Pi_S = \{\succ_{I^1} \mid I \in S\}$ be the set of the votes corresponding to S in Π_1, and let $\mathcal{E}'$ be the final election obtained from $\mathcal{E}$ by deleting all votes in Π_S. Due to the construction and the fact that S is an independent set, we have that for every $x_i \in \Gamma$ there is at most one vote in Π_S which ranks p above x_i. This implies that $N_{\mathcal{E}'}(p, x_i) \ge 2n - k + 1$ for every $x_i \in \Gamma$, and hence, p beats every $x_i \in \Gamma$ in $\mathcal{E}'$. Moreover, since q is ranked above p in every vote in Π_S, $N_{\mathcal{E}'}(p, q) = 2n-k+1$. Therefore, p beats q in $\mathcal{E}'$. Summary all above, we know that p beats every other candidate in $\mathcal{E}'$, and thus, p is the unique Copeland$^\alpha$ winner (more precisely, p is the Condorcet winner in $\mathcal{E}'$).

($\Leftarrow$:) Suppose that p becomes the unique winner after deleting at most k votes. Let Π_S be the set of the deleted votes. Let $\mathcal{E}'$ be the final election obtained from $\mathcal{E}$ by deleting all votes in Π_S. Clearly, $\mathcal{E}'$ contains at least $4n-2k+1$ votes. Since $N_{\mathcal{E}'}(q, x_i) \ge N_{\mathcal{E}}(q, x_i) - k = 3n - 2k + 2$ and $k \le n$, we known that q beats every candidate $x_i \in \Gamma$ in the final election. Since p is the unique winner in $\mathcal{E}'$, we know that Π_S contains no vote which ranks p above q (otherwise, q would also beat p, contradicting with the fact that p is the unique winner in $\mathcal{E}'$). Moreover, we know that p beats every candidate $x_i \in \Gamma$ in $\mathcal{E}'$. Since the final election contains at least $4n-2n+1$ votes and $N_{\mathcal{E}}(p, x_i) = 2n-k+2$, p beats every $x_i \in \Gamma$ if there is at most one vote in Π_S which ranks p above x_i. Due to the fact, we have the following claim.

Claim. Π_S contains no vote in Π_2.

The proof for the above claim is the same as for the claim in Theorem 4. Due to the above claim, we know that $\Pi_S \subseteq \Pi_1$. Let S be the set of 2-intervals corresponding to Π_S. Since for every $x_i \in \Gamma$ there is at most one vote in Π_S which ranks p above x_i, there is no two 2-intervals in S which intersect, implying that S is an independent set for $\mathcal{F}$. It remains to show that $|S| = k$, or equivalently, $|\Pi_S| = k$. This holds, since otherwise, q would beat every other candidate in $\mathcal{E}'$.

Now we show why the same reduction applies to CCDV-Copeland$^\alpha$-NON. We have showed above that if there is an independent set of size k, we can make p a (unique) winner. It remains to show the other direction. We begin with two observations. First, observe that we have to delete exactly k votes to make p a winner, since otherwise, q would beat every other candidate. Second, observe that q beats every candidate $x_i \in \Gamma$ in the final election no matter which k votes are deleted. Then, recall that every two 2-interval in $\mathcal{F}$ either do not intersect or they intersect at more than one point. Therefore, if we delete two votes which rank p above some candidate x_i, there must be another candidate $x_j \ne x_i$ which are ranked below p in both of the two deleted votes. This implies that p beats every candidate in $x_i \in \Gamma$ in the final election (otherwise, p would be beaten by at least two candidates in Γ, contradicting with the fact that p is a winner in $\mathcal{E}'$). However, p beats every candidate in Γ only if there is an independent set of size k for $\mathcal{F}$.

To check that the same reduction applies to DCDV-Copeland$^\alpha$-NON, observe first that no $x_i \in \Gamma$ can have a higher Copeland$^\alpha$ score than that of q in the final election—every x_i is beaten by q in the final election. Due to this, p is the only candidate which can prevent q from being a winner. The argument for CCDV-Copeland$^\alpha$-UNI then applies.

Now we consider DCDV-Copeland$^\alpha$-UNI. The reduction is similar to the above one with the difference that we create one more dummy candidate q' which lies immediately on the right side of q in the harmonious order. That is, the candidate set is $\Gamma \cup \{p, q, q'\}$ with q being the distinguished candidate, and the harmonious order is $(q, q', \vec{\Gamma}, p)$. The role of the dummy candidate q' is to guarantee that, in the final election, every candidate in Γ is beaten by both q and q'; and thus, exclude the possibility that some x_i would have a higher score than that of q in the final election. To achieve this goal, we rank q' immediately after q in every vote and remains the order of other candidates unchanged. Precisely, we create $n-1$ votes defined as $(\mathcal{L}[x_1, p], q, q')$, and $n-k+2$ votes defined as $(p, \mathcal{L}[q, x_{|\Gamma|}])$. Besides, for every 2-interval $I_i = \{I_i^1 = [x_\alpha x_{,\beta}], I_i^2 = [x_\gamma, x_\delta]\}$ with $x_\beta < x_\gamma$, we create two votes as follows.

$$(\mathcal{L}[q, q', x_\alpha), \mathcal{L}(x_\beta, x_\gamma), \mathcal{L}(x_\delta, p], \mathcal{L}[x_\alpha, x_\beta], \mathcal{L}[x_\gamma, x_\delta]);\ \text{and}$$

$$(q, q', \mathcal{L}[x_\alpha, x_\beta], \mathcal{L}[x_\gamma, x_\delta], p, \mathcal{L}[x_1, x_\alpha), \mathcal{L}(x_\beta, x_\gamma), \mathcal{L}(x_\delta, x_{|\Gamma|}]).$$

The comparisons are shown in Table 7.

	p	q	q'	x_j
p	-	$2n-k+1$	$2n-k+1$	$2n-k+2$
q	$2n$	-	$4n-k+1$	$3n-k+2$
q'	$2n$	0	-	$3n-k+2$
x_i	$2n-1$	$n-1$	$n-1$	$\times$

Table 7: Comparisons between candidates in the $\mathcal{W}[1]$-hardness proof for DCDV-Copeland$^\alpha$-UNI in Theorem 5. The comparisons between x_i and x_j cannot be exactly determined. However, the comparisons do not paly any role in the correctness proof.

We have showed in the previous proof that if there is an independent set of size k, the candidate p can prevent q from being the unique winner by deleting k votes. For the other direction, observe first that no candidate $x_i \in \Gamma$ has a chance to have a higher score than that of q since every x_i is beaten by both q and q' in the final election. Clearly, q' also has no chance to prevent q from being the unique winner since every vote ranks q above q'. Therefore, the only candidate which can prevent q from being the unique winner is p, and moreover, this happens only if p beats every candidate in Γ. The remaining argument is the same as for CCDV-Copeland$^\alpha$-UNI. □

Now we come to the Condorcet control in 4-peaked elections. Our results are summarized in the following theorem. Recall that the constructive control by deleting votes for Condorcet is $\mathcal{NP}$-hard in general, while destructive control by deleting votes is polynomial time sovlable [20].

THEOREM 6. *CCDV-Condorcet-UNI and CCDV-Condorcet-NON are $\mathcal{W}[1]$-hard in 4-peaked elections with respect to the number of deleted votes.*

PROOF. The proof for CCDV-Condorcet-UNI is exactly the same as for CCDV-Copeland$^\alpha$-UNI, and the proof for CCDV-Condorcet-NON is exactly the same as for CCDV-Copeland$^\alpha$-UNI. □

5. CONCLUSION

We have studied the complexity of the control problems in κ-peaked elections which generalize single-peaked elections by allowing at most κ-peaks in each vote. In particular, we proved that the $\mathcal{NP}$-hardness of control by adding/deleting votes in the general case remains for Condorcet, Maximin and Copeland$^\alpha$ for every $0 \leq \alpha \leq 1$ in κ-peaked elections, even when κ is equal to 3 or 4. Our reductions imply that these problems are $\mathcal{W}[1]$-hard with respect to the number of added/deleted votes. See Table 1 for a summary of our results. Several challenging and intriguing questions remain open. Among them is the complexity of control by adding/deleting votes for Condorcet, Maximin and Copeland$^\alpha$ in 2-peaked elections.

Finally, we remark that our results are based on worst-case analysis, and thus, the results do not tell us whether the problems are difficult to solve in real-world settings. In practice, it could be the case that the preferences of the voters are subjected to further restrictions in addition to the number of peaks that are allowed to appear in each vote. Nevertheless, our work pinpoints the nasty parts of the problems and is helpful in understanding and dealing with the problems in practice.

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